A Finite Element Method for Nonlinear Forced Vibrations of Rectangular Plates

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The finite element method has been extended to determine the response of large-amplitude forced vibrations of thin plates. A harmonic force matrix of a rectangular element under uniform harmonic excitation is developed for nonlinear forced vibration analysis. In-plane deformation and inertia are both considered in the formulation. Results obtained are compared with simple elliptic response, perturbation, and other approximation solutions.

Nomenclature

A	= nondimensional amplitude of vibration, w_{max}/h
B	= nondimensional forcing amplitude factor
a,b	= plate length and width
$[C],[D]$	= extensional and bending stiffness matrices
c	= ratio of volumes under plate mode shape and square of mode shape
E	= Young's modulus
$\{e\}$	= midsurface strains
F_0	= amplitude of uniform harmonic pressure
f_1,f_2	= linearizing functions
h	= plate thickness
$[k]$	= element linear stiffness matrix
$[\bar{k}]$	= element nonlinear stiffness matrix
ℓ	= length of loaded plate element
$[m]$	= element mass matrix
P_0	= nondimensional amplitude of harmonic loading
$[p]$	= element harmonic force matrix
q	= nondimensional modal displacement
t	= time
u,v,w	= displacement components in x,y,z directions, respectively
x,y,z	= Cartesian coordinates
α_i,β_i	= generalized coordinates
γ	= nondimensional nonlinearity coefficient
$\{\delta\}$	= element nodal displacements
η	= modulus of elliptic function
$\{\kappa\}$	= curvatures
λ	= circular frequency of elliptic function
ν	= Poisson's ratio
ρ	= plate mass density
τ	= nondimensional time, $t(k/m)^{1/2}$
$\{\phi\}$	= normalized mode shape
ψ	= stress function
ω	= frequency

Introduction

THIN plate structures subjected to periodic lateral loading are likely to encounter severe flexural oscillations with amplitude of the order of plate thickness. The responses predicted using the small deflection linear plate theory are no longer applicable, and, therefore, nonlinear theory taking account the effects of large amplitude has to be employed.

Following von Kármán's large deflection plate theory, the basic governing equations for the nonlinear vibration of plates were established by Herrmann.[1] Based on these equations, various approximate procedures have been investigated by numerous researchers. For example, nonlinear forced vibration of circular and rectangular plates with various boundary conditions have been studied by applying the Galerkin or Ritz method,[2-7] the Kantorovich averaging method,[8,9] various perturbation techniques,[10-13] and the incremental harmonic balance method.[14] Studies based on the simplified Berger's hypothesis[15] have also been made with the use of the Galerkin method.[16] Yamaki et al.,[7] Chia,[17] and Sathyamoorthy[18] have presented comprehensive and excellent reviews on both free and forced nonlinear vibrations of plates. Most of the studies, however, have been concerned with circular or rectangular plates due to the difficulty of the mathematical treatment. With the increased use of thin plates in many optimum- or minimum-weight designed built-up structures, it is of great importance to extend the finite element method to nonlinear forced vibration problems. In this paper, a finite element formulation is presented for the large-amplitude vibrations of thin plates subjected to harmonic loading. In-plane deformation and inertia are included in the formulation. These effects were neglected in the earlier finite element nonlinear free vibrations of plates.[19-22] A harmonic force matrix is developed for nonlinear oscillations of a rectangular plate element under uniform harmonic excitation. Formulation of the harmonic force matrix is based on the first-order mathematical approximation given by Hsu[23] that the simple harmonic forcing function $P_0\cos\omega\tau$ is simply the first-order approximation of the Fourier expansion of the Jacobian elliptic forcing function $BA\mathrm{cn}(\lambda\tau,\eta)$. And the well-known perturbation solution of the Duffing system of a simple harmonic forcing function is the first-order approximation of the simple elliptic response $A\mathrm{cn}(\lambda\tau,\eta)$. Derivation of the harmonic force and nonlinear stiffness matrices for a rectangular plate element is given. Nonlinear response to uniform and concentrated harmonic loadings and improved nonlinear free-vibration (including in-plane deformation and inertia) results are presented for rectangular plates of various boundary conditions.

Finite Element Formulation

Strain and Kinetic Energies

From von Kármán's large-deflection theory of plates, the strain-displacement relations are defined as

$$\{\epsilon\} = \{e\} + z\{\kappa\} \tag{1}$$

where the membrane or midsurface strains $\{e\}$ and cur-

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vatures $\{\kappa\}$ are given by

$$\{e\}=\begin{Bmatrix} \dfrac{\partial u}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{\partial v}{\partial y}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}+\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{2}$$

$$\{\kappa\}=\begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{3}$$

and u,v,w are displacements in the x,y,z directions, respectively.

The membrane or in-plane forces $\{N\}$ and moments $\{M\}$ are related to the strains and curvatures by

$$\{N\}=\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}=[C]\{e\} \tag{4}$$

$$\{M\}=\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}=[D]\{\kappa\} \tag{5}$$

where $[C]$ and $[D]$ are symmetric matrices of material properties. For an isotropic plate of uniform thickness h,

$$[C]=\frac{Eh}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{6}$$

$$[D]=\frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{7}$$

The total strain energy expression for a plate element can be obtained as

$$U=U_b+U_m \tag{8}$$

with

$$U_b=\tfrac{1}{2}\iint\{M\}^T\{\kappa\}\mathrm{d}x\mathrm{d}y$$

$$=\tfrac{1}{2}\iint\{\kappa\}^T[D]\{\kappa\}\mathrm{d}x\mathrm{d}y \tag{9}$$

$$U_m=\tfrac{1}{2}\iint\{N\}^T\{e\}\mathrm{d}x\mathrm{d}y$$

$$=\tfrac{1}{2}\iint\{e\}^T[C]\{e\}\mathrm{d}x\mathrm{d}y \tag{10}$$

where U_b and U_m denote the bending and membrane strain energies, respectively.

The kinetic energy of the rectangular plate element executing harmonic oscillations is

$$T=\tfrac{1}{2}\rho h\iint(\dot{u}^2+\dot{v}^2+\dot{w}^2)\mathrm{d}x\mathrm{d}y \tag{11}$$

where ρ is the mass density and the overdot means differentiation with respect to time.

Rectangular Plate Finite Element

The finite element used in the present formulation is the rectangular conforming plate element with 24 degrees of freedom due to Bogner et al.[24] The displacement functions are assumed as

$$w=\alpha_1+\alpha_2x+\alpha_3y+\alpha_4x^2+\alpha_5xy+\alpha_6y^2+\alpha_7x^3+\alpha_8x^2y$$
$$+\alpha_9xy^2+\alpha_{10}y^3+\alpha_{11}x^3y+\alpha_{12}x^2y^2+\alpha_{13}xy^3+\alpha_{14}x^3y^2$$
$$+\alpha_{15}x^2y^3+\alpha_{16}x^3y^3 \tag{12}$$

$$u=\beta_1+\beta_2x+\beta_3y+\beta_4xy \tag{13}$$

$$v=\beta_5+\beta_6x+\beta_7y+\beta_8xy \tag{14}$$

The 24 generalized coordinates

$$\{\alpha\}^T=\lfloor\alpha_1,\alpha_2,\ldots,\alpha_{16}\rfloor \tag{15}$$

$$\{\beta\}^T=\lfloor\beta_1,\beta_2,\ldots,\beta_8\rfloor \tag{16}$$

can be determined from the nodal displacements

$$\{\delta\}^T=\lfloor\{\delta_b\}^T,\{\delta_m\}^T\rfloor \tag{17}$$

with

$$\{\delta_b\}^T=\lfloor W_1,W_2,W_3,W_4,W_{x1},\ldots,W_{y1},\ldots,W_{xy1},\ldots,W_{xy4}\rfloor \tag{18}$$

$$\{\delta_m\}^T=\lfloor U_1,U_2,U_3,U_4,V_1,V_2,V_3,V_4\rfloor \tag{19}$$

The relationship between the generalized coordinates and the nodal displacements can be written as

$$\{\alpha\}=[T_b]\{\delta_b\} \tag{20}$$

$$\{\beta\}=[T_m]\{\delta_m\} \tag{21}$$

The bending strain energy U_b and the kinetic energy T lead to the element linear stiffness matrix $[k_b]$, and the element consistent mass matrices $[m_b]$ and $[m_m]$. They are given explicitly in Ref. 24.

The membrane strain energy U_m can be linearized utilizing the linearizing functions

$$\{f\}=\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}=\frac{1}{2}\begin{Bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix}=\frac{1}{2}[H]\{\alpha\}$$

$$=\frac{1}{2}[H][T_b]\{\delta_b\} \tag{22}$$

where the element displacements $\{\delta_b\}$ are obtained from the plate deflection through the iterative procedure discussed in the next section. Thus, the membrane strains become

$$\{e\} = [F]\begin{Bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix} + \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$= [F][H]\{\alpha\} + [G]\{\beta\}$$

$$= \lfloor [F][H] \quad [G] \rfloor \begin{Bmatrix} \{\alpha\} \\ \{\beta\} \end{Bmatrix} \tag{23}$$

with

$$[F] = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \\ f_2 & f_1 \end{bmatrix} \tag{24}$$

and the linearized membrane strain energy in terms of the nodal displacements is

$$U_m = \frac{1}{2} \lfloor \{\delta_b\}^T \{\delta_m\}^T \rfloor \left[\begin{bmatrix} [\bar{k}_b] & [\bar{k}_{bm}] \\ [\bar{k}_{mb}] & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [k_m] \end{bmatrix} \right] \begin{Bmatrix} \{\delta_b\} \\ \{\delta_m\} \end{Bmatrix} \tag{25}$$

where the element linear membrane stiffness matrix is given by

$$[k_m] = [T_m]^T \iint [G]^T [C][G] \mathrm{d}x\mathrm{d}y [T_m] \tag{26}$$

It is given explicitly in Ref. 24. The submatrices of the linearized nonlinear stiffness $[\bar{k}]$ are given by

$$[\bar{k}_b] = [T_b]^T \iint [H]^T [F]^T [C][F][H] \mathrm{d}x\mathrm{d}y [T_b] \tag{27}$$

$$[\bar{k}_{bm}] = [T_b]^T \iint [H]^T [F]^T [C][G] \mathrm{d}x\mathrm{d}y [T_m] \tag{28}$$

$$[\bar{k}_{mb}] = [\bar{k}_{bm}]^T \tag{29}$$

Evaluation of $[\bar{k}]$ is based on numerical integration using a four-point Gaussian integration which can exactly integrate for polynomial of cubic order.

Element Harmonic Force Matrix

In the classic continuum approach, the dynamic von Kármán equations of motion are[1]

$$\nabla^4 \psi = E(w_{,xy}^2 - w_{,xx} w_{,yy}) \tag{30}$$

$$L(w,\psi) = \rho h w_{,tt} + D\nabla^4 w - h(\psi_{,yy} w_{,xx} + \psi_{,xx} w_{,yy} - 2\psi_{,xy} w_{,xy}) - F(t) = 0 \tag{31}$$

For single-mode approximate solutions, the plate deflection is assumed as

$$w = hq(t)\phi(x,y) \tag{32}$$

where $\phi(x,y)$ satisfies the related boundary conditions. Substitution of Eq. (32) into Eq. (30), the stress function $\psi(x,y)$ is then obtained by solving the compatibility equation. Application of the Galerkin procedure $\iint L(w,\psi)\phi(x,y)\mathrm{d}x\mathrm{d}y = 0$ yields a modal equation of the Duffing form[2-6]

$$mq_{,tt} + kq + \bar{k}q^3 = F(t) \tag{33}$$

or, in nondimensional time τ, of the form

$$q_{,\tau\tau} + q + \gamma q^3 = F(\tau) \tag{34}$$

When the forcing function is a simple harmonic $F(\tau) = P_0 \cos\omega\tau$, an approximate solution of Eq. (34) using the perturbation method is the well-known result[2-4,23]

$$\left(\frac{\omega}{\omega_L}\right)^2 = 1 + \frac{3}{4}\gamma A^2 - \frac{P_0}{A} \tag{35}$$

With a simple elliptic forcing function $F(\tau) = BA\mathrm{cn}(\lambda\tau,\eta) = Bq$ as the external excitation of the Duffing system, an elliptic response[2-4,23]

$$q = A\mathrm{cn}(\lambda\tau,\eta) \tag{36}$$

is obtained as an exact solution of Eq. (34), where B is the nondimensional forcing amplitude factor, and λ and η the circular frequency and modulus of the Jacobian elliptic function. By expanding the elliptic forcing function into the Fourier series and comparing the orders of magnitude of the various harmonic components, Hsu[23] concluded that the simple harmonic forcing function $P_0\cos\omega\tau$ is the first-order approximation of the elliptic forcing function $BA\mathrm{cn}(\lambda\tau,\eta)$. He also showed that the first-order approximation of the elliptic response of Eq. (34) yields the same frequency-amplitude relations (35) as the perturbation solution. In obtaining the exact elliptic response of Eq. (34), the external excitation force $F(\tau) = Bq$ is treated as a linear spring force in the Duffing equation

$$q_{,\tau\tau} + (1-B)q + \gamma q^3 = 0 \tag{37}$$

This linear spring force Bq possesses a potential energy of $V = Bq^2/2$. The potential energy of a plate element subjected to a uniform harmonic forcing can thus be approximated by

$$V = \frac{B}{2}\iint w^2 \mathrm{d}x\mathrm{d}y \tag{38}$$

Examining Eqs. (11) and (38), the element harmonic force matrix of a plate under uniform loading $F_0\cos\omega t$ is

$$[p] = \frac{cF_0}{A\rho h^2}[m_b] \tag{39}$$

The actual applied pressure F_0 (N/m^2 or psi) is related to the dimensionless forcing parameter P_0 and the dimensionless forcing amplitude factor B by

$$B = \frac{P_0}{A} = \frac{cF_0}{A\rho h^2 \omega_L^2} \tag{40}$$

where c is a constant. For plates under uniform harmonic excitation, $c = \iint \phi \mathrm{d}x\mathrm{d}y / \iint \phi^2 \mathrm{d}x\mathrm{d}y$ which is simply the ratio of volumes under plate mode shape and the square of mode shape. The harmonic force matrix depends on the plate amplitude $A = w_{max}/h$ and P_0 (or F_0).

The amplitude of the Lagrange's equation leads to the equation of motion for the present rectangular element under the influences of inertia, elastic, large deflection, and uniform harmonic excitation forces as

$$\begin{bmatrix} [m_b] & 0 \\ 0 & [m_m] \end{bmatrix} \{\ddot{\delta}\} + \left(\begin{bmatrix} [k_b] & 0 \\ 0 & [k_m] \end{bmatrix} + \begin{bmatrix} [\bar{k}_b] & [\bar{k}_{bm}] \\ [\bar{k}_{mb}] & 0 \end{bmatrix} - \begin{bmatrix} [p] & 0 \\ 0 & 0 \end{bmatrix} \right) \{\delta\} = 0 \qquad (41)$$

The coupling between bending and membrane stretching is evident by the presence of $[\bar{k}_{bm}]$ and $[\bar{k}_{mb}]$ matrices in Eq. (41). Nonlinear free vibration is a special case of the more general nonlinear forced vibration problem with P_0 or $[p]$ = [0] in Eq. (41). Also notice the close resemblance between Eqs. (37) and (41).

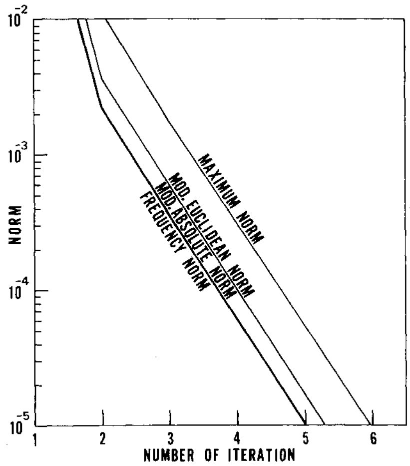

Fig. 1 Convergence characteristics.

Solution Procedures

By assembling the finite elements and applying the kinematic boundary conditions, the equations of motion for the linear free vibration of a given plate may be written as

$$\omega_L^2 [M]\{\phi\}_0 = [K]\{\phi\}_0 \qquad (42)$$

where $[M]$ and $[K]$ denote the system mass and linear stiffness matrices, respectively, ω_L the fundamental linear frequency, and $\{\phi\}_0$ the corresponding linear mode shape normalized with the maximum component to unity. The plate deflection $w_{max}\{\phi\}_0$ is then used to obtain the element nonlinear stiffness matrix $[\bar{k}]$ through Eqs. (22) and (27-29). The element harmonic force matrix is obtained through Eq. (39) for given P_0 and A. The nonlinear forced plate vibration is approximated by a linearized eigenvalue of the form

$$\omega^2 [M]\{\phi\}_1 = ([K] + [\bar{K}] - [P])\{\phi\}_1 \qquad (43)$$

where ω is the fundamental nonlinear frequency associated with amplitude A and force P_0, and $\{\phi\}_1$ the corresponding normalized mode shape of the first iteration. The iterative process can now be repeated with $w_{max}\{\phi\}_1$ until a convergence criterion is satisfied. Three displacement convergence criteria proposed by Bergan and Clough[25] and a frequency convergence criterion are used in the present study. The three displacement norms are the modified absolute norm, the modified Euclidean norm, and the maximum norm. The frequency norm is defined as $|\Delta\omega_i|/\omega_i$, where $\Delta\omega_i$ is the change in nonlinear frequency during the ith iteration cycle. A typical plot of the four norms vs number of iterations for a simply supported square plate of $a/h = 240$ with immovable in-plane edges ($u = 0$ at $x = 0$ and a, and $v = 0$ at $y = 0$ and a) subjected to a uniform harmonic force of $P_0 = 0.2$ at $A = 1.0$ is shown in Fig. 1. All four norms exhibit the important characteristics of straightness and parallelism as described in Ref. 25. Therefore, an upper bound or maximum error on displacement and frequency convergences can be estimated. The results presented in the following section, convergence is considered achieved whenever any one of the norms reaches a value of 10^{-5}.

Table 1 Free-vibration frequency ratio ω/ω_L for a simply supported plate with immovable in-plane edges

Amplitude $A = \frac{w_{max}}{h}$	Without IDI[a]: Finite element result	With in-plane deformation[4,23]: Elliptic function result	With in-plane deformation[4,23]: Peturbation solution	With IDI: Rayleigh-Ritz result[26]	With IDI: Present finite element result
		Square plate ($a/h = 240$)			
0.2	1.0185(3)[b]	1.0195	1.0196	1.0149	1.0134(3)
0.4	1.0716(3)	1.0757	1.0761	1.0583	1.0528(3)
0.6	1.1533(4)	1.1625	1.1642	1.1270	1.1154(4)
0.8	1.2565(6)	1.2734	1.2774	1.2166	1.1979(5)
1.0	1.3752(7)	1.4024	1.4097	1.3230	1.2967(6)
		Rectangular plate ($a/b = 2$, $a/h = 480$)			
0.2	1.0238(3)	1.0241	1.0241	1.0177	1.0168(3)
0.4	1.0918(4)	1.0927	1.0933	1.0690	1.0658(4)
0.6	1.1957(6)	1.1975	1.1998	1.1493	1.1439(5)
0.8	1.3264(8)	1.3293	1.3347	1.2533	1.2467(6)
1.0	1.4762(11)	1.4808	1.4903	1.3753	1.3701(8)

[a]In-plane deformation and inertia. [b]Numbers in parentheses denote number of iterations to reach a converged solution.

Table 2 **Convergence of frequency ratio with gridwork refinement for a simply supported square plate ($a/h = 240$) with immovable in-plane edges subjected to $P_0 = 0.2$**

$A = \frac{w_{max}}{h}$	Gridwork 2×2	3×3	4×4
±0.2	0.1645(3)[a]	0.1643(3)	0.1636(3)
	1.4248(3)	1.4238(3)	1.4237(3)
±0.4	0.7815(3)	0.7800(3)	0.7792(3)
	1.2697(3)	1.2682(3)	1.2677(3)
±0.6	0.9576(4)	0.9544(4)	0.9530(4)
	1.2588(4)	1.2560(4)	1.2550(4)
±0.8	1.0937(5)	1.0886(5)	1.0865(5)
	1.3026(5)	1.2981(5)	1.2963(5)
±1.0	1.2242(5)	1.2171(6)	1.2143(5)
	1.3781(5)	1.3717(6)	1.3691(5)

[a]Numbers in parentheses denote number of iterations to reach a converged solution.

Table 3 **Forced vibration frequency ratio ω/ω_L for a square plate ($a/h = 240$) with immovable in-plane edges subjected to $P_0 = 0.2$**

Amplitude $A = \frac{w_{max}}{h}$	Simple elliptic response[4,23]	Perturbation solution[4,23]	Finite element: Without IDI[a]	Finite element: With IDI
		Simply supported		
±0.2	0.1944	0.1987	0.1932(3)[b]	0.1643(3)
	1.4281	1.4281	1.4274(3)	1.4238(3)
±0.4	0.8102	0.8111	0.8052(3)	0.7800(3)
	1.2874	1.2876	1.2839(3)	1.2682(3)
±0.6	1.0084	1.0110	0.9984(4)	0.9544(4)
	1.2983	1.2995	1.2898(4)	1.2560(4)
±0.8	1.1703	1.1755	1.1528(6)	1.0886(5)
	1.3686	1.3718	1.3524(6)	1.2981(5)
±1.0	1.3283	1.3369	1.3004(7)	1.2171(6)
	1.4726	1.4789	1.4460(7)	1.3717(6)
		Clamped		
±0.2	0.1200	0.1227	0.1180(2)	0.1033(3)
	1.4195	1.4195	1.4195(2)	1.4183(3)
±0.4	0.7483	0.7485	0.7459(3)	0.7372(4)
	1.2490	1.2491	1.2477(3)	1.2426(4)
±0.6	0.8951	0.8956	0.8905(4)	0.8746(4)
	1.2117	1.2119	1.2083(4)	1.1966(4)
±0.8	0.9941	0.9954	0.9863(5)	0.9617(5)
	1.2203	1.2210	1.2137(5)	1.1938(5)
±1.0	1.0822	1.0845	1.0700(6)	1.0362(5)
	1.2540	1.2555	1.2429(6)	1.2140(5)

[a]In-plane deformation and inertia. [b]Numbers in parentheses denote number of iterations to reach a converged solution.

Results and Discussions

Improved Nonlinear Free Vibration

The fundamental frequency ratios ω/ω_L of free vibration at various amplitude $A = w_{max}/h$ for simply supported square ($a/h = 240$) and rectangular ($a/b = 2$ and $a/h = 480$) plates with immovable in-plane edges are shown in Table 1. Due to symmetry, only one-quarter of the plate modeled with nine (or 3×3 gridwork) elements of equal size is used. Both finite

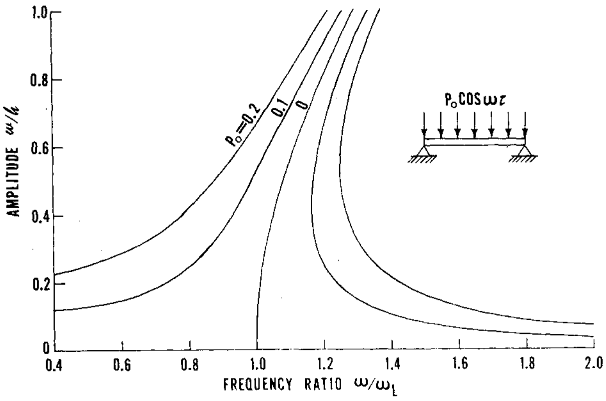

Fig. 2 **Amplitude vs frequency for a simply supported square plate ($a/h = 240$) with immovable in-plane edges under uniform loading.**

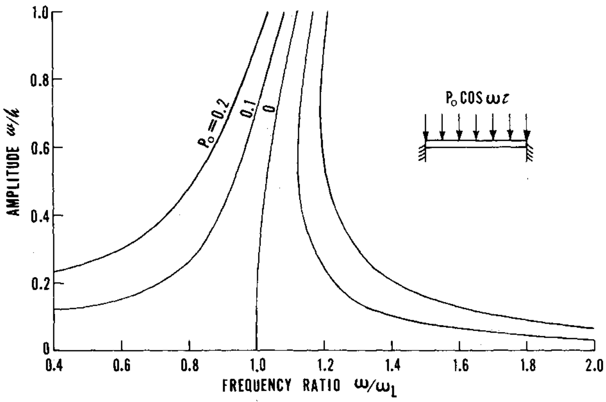

Fig. 3 **Amplitude vs frequency for a clamped square plate ($a/h = 240$) with immovable in-plane edges under uniform loading.**

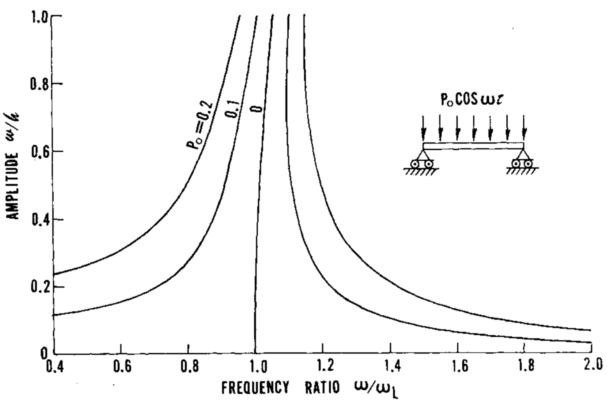

Fig. 4 **Amplitude vs frequency for a simply supported square plate ($a/h = 240$) with movable in-plane edges under uniform loading.**

element results with and without in-plane deformation and inertia (IDI) are given. It shows that the improved finite element results by including IDI in the formulation are to reduce the nonlinearity. The elliptic function and perturbation solutions[4,23] are also given to demonstrate the closeness of the earlier finite element results without IDI. Raju et al.[26] used the Rayleigh-Ritz method in their investigation of the effects of IDI on large-amplitude free-flexural vibrations of thin plates. It clearly demonstrates the remarkable agreement between the improved finite element and Rayleigh-Ritz solutions.

Convergence with Gridwork Refinement

Table 2 shows the frequency ratios of a simply supported square plate ($a/h = 240$) with immovable in-plane edges subjected to a uniform harmonic force of $P_0=0.2$ with three finite element gridwork refinements. Only one-quarter of the plate was used in the analysis due to symmetry. Examination of the results shows that the present finite element formulation exhibits excellent convergence characteristics. Therefore, a 3×3 (or nine elements) in one-quarter of the plate was used to model the plates in the remainder of the nonlinear forced responses presented, unless specified otherwise.

Nonlinear Forced Response of Plates with Immovable In-Plane Edges

Table 3 shows the frequency ratios ω/ω_L for simply supported and clamped square plates subjected to a uniform harmonic force of $P_0=0.2$. It demonstrates the closeness between the earlier finite element formulation without IDI, the simple elliptic response, and the perturbation solution (with in-plane deformation only). The present improved finite element results indicate clearly that the effects of IDI are to reduce the nonlinearity. The present finite element results of a square plate ($a/h=240$) to uniform harmonic excitation of $P_0=0$, 0.1, and 0.2 are given in Figs. 2 and 3 for simply supported and clamped boundary conditions, respectively.

Nonlinear Forced Response of Plates with Movable In-Plane Edges

The dimensionless amplitude A vs fundamental frequency ratio ω/ω_L for a simply supported square plate ($a/h=240$) with movable in-plane edges subjected to uniform harmonic load $P_0=0$, 0.1, and 0.2 is shown in Fig. 4. The nonlinearity is greatly reduced with the in-plane edges no longer restrained as compared to Fig. 2.

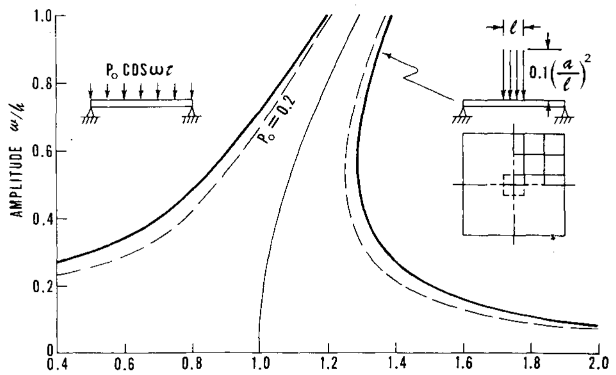

Fig. 5 Amplitude vs frequency for a simply supported square plate ($a/h=240$) with immovable in-plane edges under concentrated loading.

Concentrated Harmonic Force

The element harmonic force matrix $[p]$ given in Eq. (39) is for a uniformly distributed harmonic load. Application of the present finite element to the case of a concentrated force is to let the area of the loaded element become smaller and smaller. It is demonstrated by a concentrated force applied at the center of a simply supported square plate with immovable in-plane edges. The magnitude of the concentrated force is equal to the same plate under a uniformly distributed harmonic loading of $P_0=0.1$ ($F_0=0.66347\times10^{-2}$ psi) over the total plate area. Therefore, the uniform loading of the loaded element for the concentrated case is $F_0=0.66347\times10^{-2}(a/\ell)^2$ psi where ℓ is the length of the loaded square element. The constant c in Eq. (39), for plates subjected to concentrated force, is

$$c=\iint_{\substack{\text{(loaded}\\\text{element)}}}\phi\,\mathrm{d}x\mathrm{d}y\Big/\iint_{\substack{\text{(total}\\\text{plate}\\\text{area)}}}\phi^2\mathrm{d}x\mathrm{d}y \tag{44}$$

Table 4 gives the fundamental frequency ratios ω/ω_L at $(\ell/a)^2=16.0$, 4.0, 1.0, and 0.25%. It indicates that the convergence is rapid and $(\ell/a)^2=1.0\%$ would yield accurate frequency response. Results obtained using earlier finite elements without IDI and elliptic function (with in-plane deformation but no in-plane inertia) are also given. Nonlinear response of concentrated force obtained with $(\ell/a)^2=1.0\%$ is plotted in Fig. 5. Frequency ratios of the same plate subjected to uniform harmonic force $P_0=0.2$ is also given. It shows that the concentrated force is approximately two to three times as severe as the uniformly distributed force for the case studied.

Conclusions

1) The finite element method has been extended to treat nonlinear forced vibration problems. A harmonic force matrix is developed for a rectangular plate element subjected to uniform harmonic excitation.

2) Improved finite element results on nonlinear free-flexural vibrations of thin plates are achieved by including in-plane deformation and inertia in the formulation.

Table 4 Convergence of frequency ratio ω/ω_L with loaded area for a simply supported square plate ($a/h=240$) with immovable in-plane edges subjected to a concentrated force corresponding to $P_0=0.1\ (a/\ell)^2$ at the center

Amplitude	Elliptic function result	Finite element results at $(\ell/a)^2$, %				
		Without IDI	With IDI[a]			
$A=\frac{w_{\max}}{h}$		1	16	4	1	0.25
−0.2	1.5078	1.4097	1.4402	1.4692	1.4866	1.4957
±0.4	0.7342	0.7445	0.7652	0.7380	0.7218	0.7129
	1.3320	1.3202	1.2772	1.2940	1.3041	1.3093
±0.6	0.9688	0.9649	0.9467	0.9330	0.9254	0.9212
	1.3280	1.3148	1.2618	1.2738	1.2811	1.2849
±0.8	1.1449	1.1299	1.0839	1.0757	1.0719	1.0698
	1.3898	1.3711	1.3021	1.3115	1.3177	1.3209
±1.0	1.3103	1.2831	1.2142	1.2090	1.2076	1.2068
	1.4885	1.4606	1.3747	1.3824	1.3881	1.3910

[a]In-plane deformation and inertia.

3) Nonlinear free-vibration problem can be treated simply as a special case of the more general forced vibration by setting the harmonic force matrix equal to zero.

4) The effects of in-plane deformation and inertia are to reduce the nonlinearity slightly.

5) Concentrated loading yields responses several times as severe as the uniformly distributed load.

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